Published by Institute of Physics Publishing for SISSA/ISAS
Received: *June 9, 2004*
Accepted: *July 19, 2004*

Wrapped membranes, matrix string theory and an infinite dimensional Lie algebra

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Abstract: We examine the algebraic structure of the matrix regularization for the wrapped membrane on $\mathbb{R}^{10} \times S^1$ in the light-cone gauge. We give a concrete representation for the algebra and obtain the matrix string theory having the boundary conditions for the matrix variables corresponding to the wrapped membrane, which is referred to neither Seiberg and Sen's arguments nor string dualities. We also embed the configuration of the multi-wrapped membrane in matrix string theory.

Keywords: M-Theory, M(atrix) Theories.

http://jhep.sissa.it/archive/papers/jhep072004043/jhep072004043.pdf

Contents

1. Introduction

It is believed that the supermembrane in eleven dimensions [1] plays an important role to understand the fundamental degrees of freedom in M-theory which is a unified description of various superstring theories. Actually, the matrix-regularized theory [2, 3] of the light-cone supermembrane, which is called Matrix theory, is conjectured to describe light-cone quantized M-theory in the large-N limit [4]. Furthermore, even at finite N, Matrix theory is conjectured to describe the $p^+ = N/R$ sector of discrete light-cone quantized (DLCQ) M-theory [5].[1]

Matrix string theory [6, 7] was proposed on the heels of Matrix theory conjecture. This theory is the 1+1-dimensional U(N) super Yang-Mills theory and it is conjectured to be a non-perturbative formulation of light-cone quantized type-IIA superstring theory in the large-N limit. The theory is also conjectured to describe the $p^+ = N/R$ sector of DLCQ type-IIA superstring theory even at finite N [5]. The proposal of matrix string theory is explained, on the basis of Seiberg and Sen's arguments [8, 9], by using the T- and S-dualities with the 9-11 flip of interchanging the role of the 11th and 9th directions [6, 7].

On the other hand, type-IIA superstring in ten dimensions can be regarded as double-dimensional reduced supermembrane in eleven dimensions [10].[2] Hence, it is natural to think that matrix string theory can be regarded as the matrix-regularized theory for the wrapped supermembrane on $R^{10} \times S^1$ in the light-cone gauge. Actually, the correspondence between the wrapped supermembrane and the matrix string was given in ref. [12]. Then,

[1] In this paper we use a convention of the light-cone coordinates $x^\pm \equiv (x^0 \pm x^{10})/\sqrt{2}$. Furthermore, x^- is compactified on S^1 with radius R in DLCQ.

[2] The double-dimensional reduction was discussed classically in ref. [10]. In quantum mechanically, it is subtle whether such a reduction is realized or not [11, 12, 13].

more systematic derivation of matrix string theory by the matrix regularization of the wrapped supermembrane was presented [14]. In ref. [14], by introducing noncommutativity on the space sheet of the wrapped supermembrane, a consistent truncation of the space-sheet degrees of freedom was proposed, where it was pointed out that the underlying mathematical structure is an affine Lie algebra.

The purpose of this paper is to give a concrete matrix representation of the infinite dimensional Lie algebra in ref. [14] and obtain the matrix string theory having the boundary conditions for the matrix variables corresponding to the wrapped supermembrane. Note that the boundary conditions were assumed in ref. [12] but they are derived here. Since this method relies neither on Seiberg and Sen's arguments nor on string dualities, this gives support of the string dualities and the recovery of eleven dimensional Lorentz invariance in the large-N limit. Furthermore, we discuss the matrix regularization of the multi-wrapped supermembrane.

The plan of this paper is as follows. In the next section, we review the consistent truncation of the space-sheet degrees of freedom in the wrapped membrane theory and study the algebraic structure. In section 3, we give a concrete matrix representation for the algebra. In section 4, we obtain the matrix string theory having the boundary conditions for the matrix variables corresponding to the wrapped membrane. In section 5, we embed the configuration of the multi-wrapped membrane in matrix string theory. Final section is devoted to conclusion.

2. Consistent truncation for wrapped membrane

It is well known that Matrix theory can be obtained by truncating the infinite space-sheet degrees of freedom in the light-cone supermembrane action on R^{11} to the finite ones. On the other hand, as to the light-cone wrapped supermembrane on $\mathbb{R}^{10} \times S^1$, the truncation to finite degrees of freedom fails [12, 14]. In particular, it was pointed out that in the wrapped supermembrane action, the consistent truncation is for the target-space coordinates to take values in the representation of an affine Lie algebra [14]. In this section, we review the discussion in ref. [14].

We can truncate the degrees of freedom of the space-sheet coordinates (σ, ρ) by introducing the noncommutativity $[\sigma, \rho] = i\Theta$ (Θ: constant). This noncommutativity is encoded in the star product of functions on the space sheet,

$$f * g = f \exp\left(i\frac{1}{2}\Theta\,\epsilon^{\alpha\beta}\,\overleftarrow{\partial}_\alpha \overrightarrow{\partial}_\beta\right) g. \qquad (\alpha, \beta = \sigma, \rho)\,. \tag{2.1}$$

Then, the star-commutator for Fourier modes on the space sheet is given by[3] [16]

$$\left[e^{ik_1\sigma+ik_2\rho}, e^{ik'_1\sigma+ik'_2\rho}\right]_* = -2i\sin\left(\frac{1}{2}\Theta\, k\times k'\right)\, e^{i(k_1+k'_1)\sigma+i(k_2+k'_2)\rho}\,. \tag{2.2}$$

[3]For simplicity, we consider only toroidal membrane in this paper. Recently, the space-sheet topology in the matrix regularized membrane was discussed in ref. [15].

In the $\Theta \to 0$ limit, the space-sheet Poisson bracket is obtained,

$$\{ f, g \} = -i \lim_{\Theta \to 0} \Theta^{-1} \, [f, g]_* \, . \tag{2.3}$$

Henceforth, we set $\Theta = 4\pi/N$ ($N = 2M + 1$: odd number). Then, the Fourier modes $e^{ipN\sigma}$, $e^{irN\rho}$ ($p, r \in \mathbb{Z}$) commute with any modes and hence they are central elements in the star-commutator algebra. This means that they can be consistently modded out from the star-commutator algebra, since left and right multiplications coincide on any modes. Thus we can identify them with the identity operator and obtain the following equivalence relation,

$$e^{i(k_1+pN)\sigma+ik_2\rho} \approx e^{ik_1\sigma+ik_2\rho} \, , \tag{2.4}$$
$$e^{ik_1\sigma+i(k_2+rN)\rho} \approx e^{ik_1\sigma+ik_2\rho} \, . \tag{2.5}$$

Under the identification, we can truncate the infinite dimensional algebra to the finite dimensional algebra $\mathfrak{u}(N)$ consistently. Then, the mode numbers of $e^{ik_1\sigma+ik_2\rho}$ are restricted to $k_1, k_2 = 0, \pm 1, \pm 2, \ldots, \pm M$. If we adopt such a consistent truncation for the light-cone supermembrane on R^{11}, we can obtain Matrix theory.

In the case of the wrapped membrane, we need to add a linear function ρ representing the wrapping to the generators of the star-commutator algebra. Then the star commutators are given by eq. (2.2) and

$$\left[\rho, e^{ik_1\sigma+ik_2\rho} \right]_* = \frac{4\pi k_1}{N} \, e^{ik_1\sigma+ik_2\rho} \, . \tag{2.6}$$

Thus, in this case, we cannot truncate this star-commutator algebra to a finite dimensional one because the star commutator $\left[\rho, e^{ipN\sigma} \right]_* = 4\pi p e^{ipN\sigma}$ indicates that $e^{ipN\sigma}$ cannot be the central elements and hence the equivalence (2.4) is not valid. On the other hand, $e^{iqN\rho}$ are the central elements and the equivalence (2.5) is still valid. Then, we can truncate only the Fourier modes with respect to ρ and the truncated generators are given by $\{ e^{ik_1\sigma+ik_2\rho}, \rho \,|\, k_1 = 0, \pm 1, \pm 2, \ldots, \pm\infty, \, k_2 = 0, \pm 1, \pm 2, \ldots, \pm M \}$ [14]. Note that although we cannot identify $e^{ipN\sigma}$ with the central elements, they form an ideal of the truncated star-commutator algebra. Hence this algebra is not simple and henceforth we restrict to the quotient by this ideal in this section. In the next section, we will comment on the ideal.

Although this quotient is infinite dimensional, the rank is finite. Actually, we can adopt N generators $\{ e^{ik\rho}, \rho \,|\, k = \pm 1, \pm 2, \ldots, \pm M \}$ as the Cartan subalgebra generators. We take the basis of the Cartan subalgebra generators as follows,

$$H^k = \frac{1}{N} \sum_{l=-M,\, l\neq 0}^{M} \lambda^{kl} (\lambda^{-l} - \lambda^{l}) \, e^{il\rho} \, , \qquad (k = 0, \pm 1, \pm 2, \ldots, \pm M) \tag{2.7}$$

$$D = \frac{1}{4\pi} \rho - \frac{1}{N} \sum_{l=-M,\, l\neq 0}^{M} \frac{1}{\lambda^{-l} - \lambda^{l}} \, e^{il\rho} \, , \tag{2.8}$$

where $\lambda \equiv e^{2\pi i/N}$. Note that λ has the following property,

$$\sum_{l=-M}^{M} \lambda^{kl} = N\delta_{k,0}^{(N)}\,, \tag{2.9}$$

where the indices of the Kronecker symbol $\delta_{k,l}^{(N)}$ are understood to be modulo N. In eq. (2.7), the index k runs from $-M$ to M. Thus, at first sight the number of the generators in eq. (2.7) seems to be N $(= 2M+1)$. However, the number of the independent generators is $N-1$. Actually, $H^0 = -\sum_{l=-M,\, l\neq 0}^{M} H^l$ due to eq. (2.9) and hence H^0 is not independent. By using eq. (2.9), eqs.(2.7) and (2.8) are rewritten by

$$e^{ik\rho} = \frac{1}{\lambda^{-k}-\lambda^{k}} \sum_{l=-M}^{M} \lambda^{-kl} H^l, \quad (k = \pm 1, \pm 2, \ldots, \pm M) \tag{2.10}$$

$$\rho = 4\pi D + \frac{4\pi}{N} \sum_{k=-M}^{M} \left\{ \sum_{l=-M,\, l\neq 0}^{M} \frac{\lambda^{-kl}}{(\lambda^{-l}-\lambda^{l})^2} \right\} H^k\,. \tag{2.11}$$

As for the remaining infinite raising and lowering generators, we take the following basis,

$$E^k_{pN+q} = \frac{1}{N} \sum_{l=-M}^{M} \lambda^{kl}\, e^{i(pN+q)\sigma + il\rho}\,, \qquad (p = 0, \pm 1, \pm 2, \ldots) \tag{2.12}$$

$$E^k_{pN} = \frac{1}{N} \sum_{l=-M,\, l\neq 0}^{M} \lambda^{kl}(\lambda^{-l}-\lambda^{l})\, e^{ipN\sigma + il\rho}\,, \qquad (p = \pm 1, \pm 2, \ldots) \tag{2.13}$$

where $k = 0, \pm 1, \ldots, \pm M$, $q = \pm 1, \pm 2, \ldots, \pm M$. Note that the following relations hold,

$$E^0_{pN} = -\sum_{l=-M,\, l\neq 0}^{M} E^l_{pN}\,, \qquad (E^k_{pN})^\dagger = E^k_{-pN}\,, \qquad (E^k_{pN+q})^\dagger = E^k_{-(pN+q)}\,. \tag{2.14}$$

Eqs. (2.12) and (2.13) are inverted as follows,

$$e^{i(pN+q)\sigma + ik\rho} = \sum_{l=-M}^{M} \lambda^{-kl} E^l_{pN+q}\,, \qquad (k = 0, \pm 1, \pm 2, \ldots, \pm M) \tag{2.15}$$

$$e^{ipN\sigma + ik\rho} = \frac{1}{\lambda^{-k}-\lambda^{k}} \sum_{l=-M}^{M} \lambda^{-kl} E^l_{pN}\,. \quad (k = \pm 1, \pm 2, \ldots, \pm M) \tag{2.16}$$

From eqs. (2.2) and (2.6), we obtain the following commutators for the generators (2.7), (2.8), (2.12) and (2.13), although the calculation is a bit lengthly,

$$\Big[H^k, H^l\Big]_* = 0\,, \tag{2.17}$$

$$\Big[H^k, D\Big]_* = 0\,, \tag{2.18}$$

$$\Big[H^k, E^l_{pN}\Big]_* = 0\,, \tag{2.19}$$

$$\left[H^k, E^l_{pN+q}\right]_* = \left(\delta^{(N)}_{k-l+q-1,0} - \delta^{(N)}_{k-l+q+1,0} - \delta^{(N)}_{k-l-q-1,0} + \delta^{(N)}_{k-l-q+1,0}\right) E^l_{pN+q}\,, \tag{2.20}$$

$$\left[D, E^l_{pN}\right]_* = pE^l_{pN}\,, \tag{2.21}$$

$$\left[D, E^l_{pN+q}\right]_* = \omega(l, pN+q)\, E^l_{pN+q}\,, \qquad \left(\omega(l, pN+q) \equiv p + \mathrm{sgn}(q) \sum_{s=0}^{|q|-1} \delta^{(N)}_{2s,\, l+|q|-1}\right) \tag{2.22}$$

$$\left[E^k_{pN}, E^l_{rN}\right]_* = 0\,, \tag{2.23}$$

$$\left[E^k_{pN}, E^l_{rN+s}\right]_* = \left(\delta^{(N)}_{k-l+s-1,0} - \delta^{(N)}_{k-l+s+1,0} - \delta^{(N)}_{k-l-s-1,0} + \delta^{(N)}_{k-l-s+1,0}\right) E^l_{(p+r)N+s}\,, \tag{2.24}$$

$$\left[E^k_{pN+q}, E^l_{rN+s}\right]_* = \begin{cases} \mathrm{sgn}(q) \displaystyle\sum_{t=0}^{|q|-1} H^{2t+k-|q|+1}\,, & (k=l, q+s=0, p+r=0) \\ \mathrm{sgn}(q) \displaystyle\sum_{t=0}^{|q|-1} E^{2t+k-|q|+1}_{(p+r)N}\,, & (k=l, q+s=0, p+r\neq 0) \\ E^{k+s}_{(p+r)N+q+s}\,, & \begin{pmatrix} k-l+q+s=0 & \mathrm{mod}\ N \\ k-l-q-s\neq 0 & \mathrm{mod}\ N \end{pmatrix} \\ -E^{k-s}_{(p+r)N+q+s}\,, & \begin{pmatrix} k-l+q+s\neq 0 & \mathrm{mod}\ N \\ k-l-q-s=0 & \mathrm{mod}\ N \end{pmatrix} \\ 0\,. & \text{otherwise}\,. \end{cases} \tag{2.25}$$

From these star commutators, we can obtain the root system of the quotient. For simplicity, we consider the $N=3$ $(M=1)$ case, first. The root system is given in figure 1, where we have changed the basis of the Cartan subalgebra generators $\{H^k, D | k = \pm 1\} \to \{H_\pm, D\}$ $(H^{\pm 1} = H_+ \pm \sqrt{3}H_-)$.[4] In figure 1, we see the infinite series of the subalgebra $\mathfrak{su}(3)$ in the direction of D, where D is a derivation. In this root system, the nine generators commute with D: One of them is D itself and the remaining eight generators are $H^{\pm 1}$, $E^{\pm 1}_{\pm 1}$ and $E^0_{\pm 2}$ which constitute the zero-mode subalgebra of the loop algebra over $\mathfrak{su}(3)$. Thus this root system agrees with that of affine $\mathfrak{su}(3)$ *except for a central element.*[5] In the zero-mode subalgebra, root vectors α_1, α_2 corresponding to the generators $E^{\pm 1}_1$ are simple roots.

[4] H^k $(k=\pm 1)$ are the Cartan subalgebra generators in the Chevalley basis of $\mathfrak{su}(3)$, while $H_\pm$ are those in the ordinary Cartan-Weyl basis of $\mathfrak{su}(3)$.

[5] With a finite dimensional simple Lie algebra $\bar{\mathfrak{g}}$, the (untwisted) affine Lie algebra $\mathfrak{g}$ is obtained by extending $\bar{\mathfrak{g}}$ to the infinite dimensional Lie algebra $\bar{\mathfrak{g}}_{\mathrm{loop}}$ of the loop algebra, centrally extending $\bar{\mathfrak{g}}_{\mathrm{loop}}$ and adding a derivation D, i.e., $\mathfrak{g} = \bar{\mathfrak{g}}_{\mathrm{loop}} \oplus \mathbb{C}K \oplus \mathbb{C}D$, where K is a central element (See e.g., ref. [17]). Thus, in order for this root system to agree with that of an affine Lie algebra, we need to centrally extend the star commutators (2.2) and (2.6) (or (2.17)–(2.25)). The central extension would be related to anomaly in the supermembrane.

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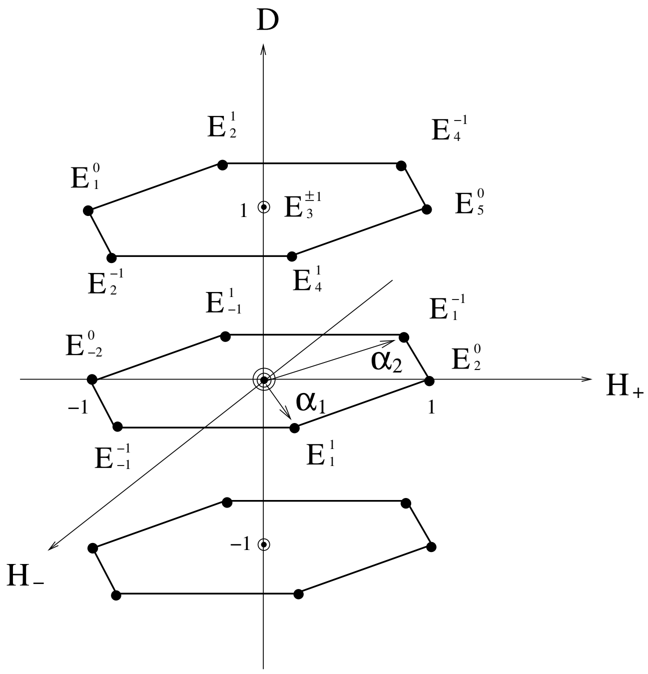

Figure 1: The root system in the $N = 3$ case. Here, $\{H_\pm, D\}$ are the Cartan subalgebra generators. α_1, α_2 are simple roots of the zero-mode subalgebra.

generators		number
H^l	$(l \neq 0)$	$2M$
$E^l_{\pm 1}$	$(l \neq 0)$	$2M \times 2$
$E^l_{\pm 2}$	$(l \neq \pm 1)$	$(2M - 1) \times 2$
$E^l_{\pm 3}$	$(l \neq \pm 2, 0)$	$(2M - 2) \times 2$
$\vdots$		$\vdots$
$E^l_{\pm(M-1)}$	$(l \neq \pm(M-2), \pm(M-4), \ldots)$	$(M + 2) \times 2$
$E^l_{\pm M}$	$(l \neq \pm(M-1), \pm(M-3), \ldots)$	$(M + 1) \times 2$
$E^l_{\pm(M+1)}$	$(l = \pm(M-1), \pm(M-3), \ldots)$	$M \times 2$
$E^l_{\pm(M+2)}$	$(l = \pm(M-2), \pm(M-4), \ldots)$	$(M - 1) \times 2$
$\vdots$		$\vdots$
$E^l_{\pm(N-2)}$	$(l = \pm 1)$	2×2
$E^l_{\pm(N-1)}$	$(l = 0)$	1×2
	total	$N^2 - 1$

Table 1: The list of generators which constitute the zero-mode subalgebra of the loop algebra over $\mathfrak{su}(N)$.

The analysis of the general N case is performed similarly. From eqs. (2.18), (2.21) and (2.22), we see N^2 generators which commute with D. Among them, $N^2 - 1$ generators (see table 1) constitute the zero-mode subalgebra of the loop algebra over $\mathfrak{su}(N)$ and the

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remaining one is D itself. Thus this root system agrees with that of affine $\mathfrak{su}(N)$ except for a central element. In the zero-mode subalgebra, $N-1$ root vectors corresponding to the generators E_1^l $(l = \pm 1, \pm 2, \ldots, \pm M)$ are simple roots. Actually, from the star commutators (2.17)–(2.25), $3(N-1)$ generators $\{E_{\pm 1}^k, H^l \,|\, k, l = \pm 1, \pm 2, \ldots, \pm M\}$ satisfy the Chevalley-Serre relations of $\mathfrak{su}(N)$ (see e.g., ref. [17]),

$$\left[H^k, H^l\right]_* = 0\,, \tag{2.26}$$

$$\left[H^k, E_{\pm 1}^l\right]_* = \pm(2\delta_{k,l}^{(N)} - \delta_{k-l,-2}^{(N)} - \delta_{k-l,2}^{(N)})\, E_{\pm 1}^l = \pm A^{lk} E_{\pm 1}^l\,, \tag{2.27}$$

$$\left[E_{+1}^k, E_{-1}^l\right]_* = \delta_{k,l} H^k\,, \tag{2.28}$$

$$(\mathrm{ad}_{E_{\pm 1}^k})^{1-A^{lk}}(E_{\pm 1}^l) = 0\,, \qquad (k \neq l)\,, \tag{2.29}$$

where A^{lk} is the (l,k)-component of the Cartan matrix of $\mathfrak{su}(N)$ and $\mathrm{ad}_x(y) \equiv [x, y]$.

3. Representation of the algebra

In this section, we give a concrete representation of the star-commutator algebra in the previous section. Actually, we can represent the generators (2.7)–(2.8) and (2.12)–(2.13), which satisfy eqs.(2.17)–(2.25), as the $N \times N$ matrices with a continuous parameter θ,

$$H^k \to \left(\left(H^k\right)_{ab}\right), \qquad (H^k)_{ab} \equiv \delta_{ab}\,(\delta_{k,2a}^{(N)} - \delta_{k,2(a-1)}^{(N)})\,, \tag{3.1}$$

$$D \to \left(\frac{-i}{2}\,\partial_\theta\,\delta_{ab}\right), \tag{3.2}$$

$$E_{pN}^k \to \left((E_{pN}^k)_{ab}\right), \qquad (E_{pN}^k)_{ab} \equiv e^{i2p\theta}(H^k)_{ab}\,, \tag{3.3}$$

$$E_{pN+q}^k \to \left((E_{pN+q}^k)_{ab}\right), \qquad (E_{pN+q}^k)_{ab} \equiv e^{i2\omega(k,pN+q)\theta}\,\delta_{b-a,q}^{(N)}\,\delta_{a+b,k+1}^{(N)}\,, \tag{3.4}$$

where matrix indices $a, b = 1, 2, \ldots, N$. It is easy to see that these matrices satisfy the star commutators (2.17)–(2.25). Furthermore, by using the above matrices and eqs. (2.10)–(2.11) and (2.15)–(2.16), the matrix representations of the linear function ρ and Fourier modes $e^{ipN\sigma+ik\rho}$ $(k = \pm 1, \pm 2, \ldots, \pm M,\ p \in \mathbb{Z})$, $e^{i(pN+q)\sigma+ik\rho}$ $(k = 0, \pm 1, \pm 2, \ldots, \pm M,\ q = \pm 1, \pm 2, \ldots, \pm M,\ p \in \mathbb{Z})$ are given by

$$\rho \to -2\pi i\partial_\theta \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} + 4\pi \begin{pmatrix} \frac{M}{N} & & & & 0 \\ & \frac{M-1}{N} & & & \\ & & \frac{M-2}{N} & & \\ & & & \ddots & \\ 0 & & & & -\frac{M}{N} \end{pmatrix} \tag{3.5}$$

$$
e^{ipN\sigma+ik\rho} \to \tau^p \lambda^{-k} \begin{pmatrix} 1 & & & & \\ & \lambda^{-2k} & & 0 & \\ & & \lambda^{-4k} & & \\ & 0 & & \ddots & \\ & & & & \lambda^{-2(N-1)k} \end{pmatrix}, \tag{3.6}
$$

$$
e^{i(pN+q)\sigma+ik\rho} \to \begin{cases} \tau^p \lambda^{-k(q+1)} \\ \times \begin{pmatrix} \overbrace{0 \quad \cdots \quad 0}^{q} & 1 & & & \\ & 0 & \lambda^{-2k} & & \\ & & \ddots & \ddots & \\ & & & 0 & \lambda^{-2(N-q-1)k} \\ \lambda^{-2(N-q)k}\tau & & & & 0 \\ \ddots & & & & \vdots \\ \underbrace{0 \quad \cdots \quad 0}_{q-1} \ \lambda^{-2(N-1)k}\tau & 0 & & & 0 \end{pmatrix}, \ (q>0) \\ \\ \tau^{p-1} \lambda^{-k(q+1)} \\ \times \begin{pmatrix} \overbrace{0 \quad \cdots \quad 0}^{N+q} & 1 & & & \\ & 0 & \lambda^{-2k} & & \\ & & \ddots & \ddots & \\ & & & 0 & \lambda^{-2(-q-1))k} \\ \lambda^{-2(-q)k}\tau & & & & 0 \\ \ddots & & & & \vdots \\ \underbrace{0 \quad \cdots \quad 0}_{N+q-1} \ \lambda^{-2(N-1)k}\tau & 0 & & & 0 \end{pmatrix}, \ (q<0) \end{cases} \tag{3.7}
$$

where $\tau \equiv e^{2i\theta}$. So far, we have concentrated on the quotient by the ideal $\{e^{ipN\sigma}\}$ $(p \in \mathbb{Z})$. However, it is easy to extend the discussion in the previous section with the ideal included. Then the matrix representations of the generators are given by

$$
e^{ipN\sigma} \to \tau^p \begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}. \tag{3.8}
$$

We summarize the matrix representations of the Fourier modes (3.6)–(3.8) as follows,

$$
e^{i(pN+q)\sigma+ik\rho} \to \begin{cases}
\tau^{p}\lambda^{-k(q+1)} \\
\times \begin{pmatrix}
\overbrace{0 \quad \cdots \quad 0}^{q} & 1 & & & \\
 & 0 & \lambda^{-2k} & & \\
 & & \ddots & \ddots & \\
 & & & 0 & \lambda^{-2(N-q-1)k} \\
\lambda^{-2(N-q)k}\tau & & & & 0 \\
 & \ddots & & & \vdots \\
\underbrace{0 \quad \cdots \quad 0}_{q-1} \ \lambda^{-2(N-1)k}\tau & 0 & & & 0
\end{pmatrix}, \ (q>0) \\[2ex]
\tau^{p}\lambda^{-k} \begin{pmatrix}
1 & & & & \\
 & \lambda^{-2k} & & 0 & \\
 & & \lambda^{-4k} & & \\
 & 0 & & \ddots & \\
 & & & & \lambda^{-2(N-1)k}
\end{pmatrix}, \qquad (q=0) \\[2ex]
\tau^{p-1}\lambda^{-k(q+1)} \\
\times \begin{pmatrix}
\overbrace{0 \quad \cdots \quad 0}^{N+q} & 1 & & & \\
 & 0 & \lambda^{-2k} & & \\
 & & \ddots & \ddots & \\
 & & & 0 & \lambda^{-2(-q-1))k} \\
\lambda^{-2(-q)k}\tau & & & & 0 \\
 & \ddots & & & \vdots \\
\underbrace{0 \quad \cdots \quad 0}_{N+q-1} \ \lambda^{-2(N-1)k}\tau & 0 & & & 0
\end{pmatrix}, \ (q<0)
\end{cases} \tag{3.9}
$$

where $k, q = 0, \pm 1, \pm 2, \ldots, \pm M$, $p \in \mathbb{Z}$.

4. From wrapped membrane to matrix string

In this section, we show that the consistent truncation of the light-cone wrapped supermembrane on $R^{10} \times S^1$ leads to matrix string theory. In particular, by using the matrix representations in the previous section, we can derive the boundary conditions of the matrix variables corresponding to the wrapped supermembrane.

Our starting point is the action of the light-cone wrapped supermembrane on $R^{10} \times S^1$,[6] (Here we just write it only with the bosonic degrees of freedom. Fermions are straightfor-

[6]Precisely speaking, when the membrane has the non-trivial space-sheet topology, we need to impose the global constraints to the action (4.1) [18]. However, for simplicity, such constraints are ignored in this paper.

wardly included.)

$$S_{\rm WM} = \frac{LT}{2}\int d\tau \int_0^{2\pi} d\sigma d\rho \left[(D_\tau X^i)^2 - \frac{1}{2L^2}\{X^i, X^j\}^2\right], \tag{4.1}$$

$$D_\tau X^i = \partial_\tau X^i - \frac{1}{L}\{A, X^i\}, \tag{4.2}$$

where $i, j = k, 9\,(k = 1, \ldots, 8)$ and L is a radius of the target space S^1. We take X^9 as the S^1 direction in the action and X^i, A are Fourier expanded as

$$\begin{aligned} X^9 &= wL\rho + Y \\ &= wL\rho + \sum_{k_1,k_2=-\infty}^{\infty} Y_{(k_1,k_2)}\, e^{ik_1\sigma + ik_2\rho}\,, \end{aligned} \tag{4.3}$$

$$X^k = \sum_{k_1,k_2=-\infty}^{\infty} X^k_{(k_1,k_2)}\, e^{ik_1\sigma + ik_2\rho}\,, \tag{4.4}$$

$$A = \sum_{k_1,k_2=-\infty}^{\infty} A_{(k_1,k_2)}\, e^{ik_1\sigma + ik_2\rho}\,, \tag{4.5}$$

where $w(\neq 0)$ is a wrapping number. Now we introduce the noncommutativity $[\sigma, \rho] = 4\pi i/N$ on the space-sheet and carry out the consistent truncation,

$$\begin{aligned} X^9 &= wL\rho + \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-M}^{M} Y_{(k_1,k_2)}\, e^{ik_1\sigma+ik_2\rho} \\ &= wL\rho + \sum_{p=-\infty}^{\infty}\sum_{q=-M}^{M}\sum_{k=-M}^{M} Y_{(pN+q,k)}\, e^{i(pN+q)\sigma+ik\rho}\,, \end{aligned} \tag{4.6}$$

$$X^k = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-M}^{M} X^k_{(k_1,k_2)}\, e^{ik_1\sigma+ik_2\rho} = \sum_{p=-\infty}^{\infty}\sum_{q=-M}^{M}\sum_{k=-M}^{M} X^k_{(pN+q,k)}\, e^{i(pN+q)\sigma+ik\rho}\,, \tag{4.7}$$

$$A = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-M}^{M} A_{(k_1,k_2)}\, e^{ik_1\sigma+ik_2\rho} = \sum_{p=-\infty}^{\infty}\sum_{q=-M}^{M}\sum_{k=-M}^{M} A_{(pN+q,k)}\, e^{i(pN+q)\sigma+ik\rho}\,. \tag{4.8}$$

By using eq. (3.9), the truncated Fourier modes are represented by $N \times N$ matrices with a continuous parameter θ ($X = Y, X^k, A$),

$$\begin{aligned} X &= \sum_{p=-\infty}^{\infty}\sum_{q=-M}^{M}\sum_{k=-M}^{M} X_{(pN+q,k)}\, e^{i(pN+q)\sigma+ik\rho} \\ &\to X(\theta) = \sum_{p=-\infty}^{\infty}\sum_{q=-M}^{M}\sum_{k=-M}^{M} X_{(pN+q,k)}\, \tau^p\, \lambda^{-k(q+1)} M_q^k\,, \end{aligned} \tag{4.9}$$

where

$$
M_q^k = \begin{cases}
\begin{pmatrix}
\overbrace{0 \quad \cdots \quad 0}^{q} & 1 & & & & \\
 & 0 & \lambda^{-2k} & & & \\
 & & \ddots & \ddots & & \\
 & & & 0 & \lambda^{-2(N-q-1)k} & \\
\lambda^{-2(N-q)k}\tau & & & & 0 & \\
\quad\ddots & & & & \vdots & \\
\underbrace{0 \quad \cdots \quad 0}_{q-1} \ \lambda^{-2(N-1)k}\tau & 0 & & & 0 &
\end{pmatrix}, & (q > 0) \\[2ex]
\begin{pmatrix}
1 & & & & \\
 & \lambda^{-2k} & & 0 & \\
 & & \lambda^{-4k} & & \\
 & 0 & & \ddots & \\
 & & & & \lambda^{-2(N-1)k}
\end{pmatrix}, & (q = 0) \\[2ex]
\begin{pmatrix}
\overbrace{0 \quad \cdots \quad 0}^{N+q} & \tau^{-1} & & & & \\
 & 0 & \lambda^{-2k}\tau^{-1} & & & \\
 & & \ddots & \ddots & & \\
 & & & 0 & \lambda^{-2(-q-1))k}\tau^{-1} & \\
\lambda^{-2(-q)k} & & & & 0 & \\
\quad\ddots & & & & \vdots & \\
\underbrace{0 \quad \cdots \quad 0}_{N+q-1} \ \lambda^{-2(N-1)k} & 0 & & & 0 &
\end{pmatrix}. & (q < 0)
\end{cases} \tag{4.10}
$$

As an example, we give the matrix representation for the case of $N = 3$ explicitly,

$$
X^{(N=3)}(\theta) = \sum_{p=-\infty}^{\infty} \sum_{k=-1}^{1} \tau^p \begin{pmatrix} X_{(3p,k)}\lambda^{-k} & X_{(3p+1,k)}\lambda^{-2k} & \tau^{-1}X_{(3p-1,k)} \\ X_{(3p-1,k)}\lambda^{-2k} & X_{(3p,k)}\lambda^{-3k} & X_{(3p+1,k)}\lambda^{-4k} \\ \tau X_{(3p+1,k)} & X_{(3p-1,k)}\lambda^{-4k} & X_{(3p,k)}\lambda^{-5k} \end{pmatrix}. \tag{4.11}
$$

We consider the double-dimensional reduction from the wrapped supermembrane on $R^{10} \times S^1$ to type-IIA superstring on R^{10}. Classically, this is to remove the non-zero Fourier modes with respect to ρ by hand [10]. After such a reduction, the matrix representation $X(\theta)|_{\mathrm{DDR}}$ is given by

$$
X(\theta)|_{\mathrm{DDR}} = \sum_{p=-\infty}^{\infty} \sum_{q=-M}^{M} X_{(pN+q,0)} \tau^p M_q^0 \,. \tag{4.12}
$$

In this case also, we just give the matrix representation in the $N = 3$ case,

$$
X^{(N=3)}(\theta)|_{\mathrm{DDR}} = \sum_{p=-\infty}^{\infty} \tau^p \begin{pmatrix} X_{(3p,0)} & X_{(3p+1,0)} & \tau^{-1}X_{(3p-1,0)} \\ X_{(3p-1,0)} & X_{(3p,0)} & X_{(3p+1,0)} \\ \tau X_{(3p+1,0)} & X_{(3p-1,0)} & X_{(3p,0)} \end{pmatrix}. \tag{4.13}
$$

Note that $X(\theta)|_{\rm DDR}$ is *not* a diagonal matrix, even though only the zero-modes w.r.t. ρ have been extracted in $X(\theta)|_{\rm DDR}$. This matrix is represented in the basis where the zero modes w.r.t. σ are placed diagonally. From the physical point of view, however, since the zero modes w.r.t. ρ are identified with the coordinates of type-IIA superstring, the basis where the zero modes with respect to ρ are placed diagonally seems to be natural. Actually, in ref. [12], in the latter basis, a correspondence of the wrapped supermembrane with matrix string was discussed. Hence we diagonalize the matrix (4.12). Actually, we can diagonalize it with the following unitary matrix P,

$$P = TS\,, \tag{4.14}$$

$$T \equiv \begin{pmatrix} \tau^{-\frac{M}{N}} & & & & 0 \\ & \tau^{-\frac{(M-1)}{N}} & & & \\ & & \tau^{-\frac{(M-2)}{N}} & & \\ & 0 & & \ddots & \\ & & & & \tau^{\frac{M}{N}} \end{pmatrix}, \tag{4.15}$$

$$S \equiv \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & & 1 \\ 1 & \lambda^2 & \lambda^4 & \cdots & \lambda^{2(N-1)} \\ 1 & \lambda^4 & \lambda^8 & \cdots & \lambda^{4(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda^{2(N-1)} & \lambda^{4(N-1)} & \cdots & \lambda^{2(N-1)^2} \end{pmatrix}. \tag{4.16}$$

Then we have

$$\begin{aligned} P^\dagger X(\theta)|_{\rm DDR} P &= S^\dagger T^\dagger\, X(\theta)|_{\rm DDR}\, TS \\ &= S^\dagger \left(\sum_{p=-\infty}^{\infty} \sum_{q=-M}^{M} X_{(pN+q,0)}\, \tau^{p+\frac{q}{N}}\, V^q \right) S \\ &= \sum_{p=-\infty}^{\infty} \sum_{q=-M}^{M} X_{(pN+q,0)} \tau^{p+\frac{q}{N}} U^q \\ &= \sum_{p=-\infty}^{\infty} \sum_{q=-M}^{M} X_{(pN+q,0)}\, \tau^{p+\frac{q}{N}} \begin{pmatrix} 1 & & & & & 0 \\ & \lambda^{2q} & & & & \\ & & \lambda^{4q} & & & \\ & & & \lambda^{6q} & & \\ & 0 & & & \ddots & \\ & & & & & \lambda^{2(N-1)q} \end{pmatrix} \\ &\equiv \begin{pmatrix} x_1(\theta) & & & & & 0 \\ & x_2(\theta) & & & & \\ & & x_3(\theta) & & & \\ & & & x_4(\theta) & & \\ & 0 & & & \ddots & \\ & & & & & x_N(\theta) \end{pmatrix}, \end{aligned} \tag{4.17}$$

where U and V are the clock and shift matrices, respectively,

$$U = \begin{pmatrix} 1 & & & & & 0 \\ & \lambda^2 & & & & \\ & & \lambda^4 & & & \\ & & & \lambda^6 & & \\ & 0 & & & \ddots & \\ & & & & & \lambda^{2(N-1)} \end{pmatrix}, \tag{4.18}$$

$$V = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ \vdots & & & \ddots & \ddots & \\ 0 & & & & 0 & 1 \\ 1 & 0 & & \cdots & & 0 \end{pmatrix}. \tag{4.19}$$

U and V satisfy $U^N = V^N = 1$. For $q < 0$, $U^q \equiv (U^\dagger)^{-q}$, $V^q \equiv (V^\dagger)^{-q}$ and $S^\dagger V S = U$, $S^\dagger U S = V^{-1}$. The diagonal elements $x_a(\theta)$ (4.17) in matrix string theory are expressed by the Fourier coefficients in the wrapped supermembrane theory,

$$\begin{aligned} x_a(\theta) &= \sum_{p=-\infty}^{\infty} \sum_{q=-M}^{M} X_{(pN+q,0)}\, e^{2i(p+\frac{q}{N})\theta}\, e^{i\frac{4(a-1)\pi}{N}q} \\ &= \sum_{n=-\infty}^{\infty} X_{(n,0)}\, e^{2i\frac{n}{N}(\theta+2(a-1)\pi)}. \end{aligned} \tag{4.20}$$

Then it is easy to see that these diagonal elements satisfy the following boundary conditions,

$$x_a(\theta+2\pi) = x_{a+1}(\theta)\,, \qquad (a = 1, \ldots, N-1) \tag{4.21}$$

$$x_N(\theta+2\pi) = x_1(\theta)\,. \tag{4.22}$$

Thus we have derived that via the double-dimensional reduction, the wrapped supermembrane corresponds to a long string, which is given by the boundary conditions (4.21)-(4.22), in matrix string theory.

Next, we consider the k-th Fourier mode ($k > 0$) with respect to ρ,

$$X(\theta)|_{k\text{-th}} = \sum_{p=-\infty}^{\infty} \sum_{q=-M}^{M} X_{(pN+q,k)}\, \tau^p\, \lambda^{-k(q+1)} M_q^k\,. \tag{4.23}$$

By using the same unitary matrix P (4.14), we represent this matrix in the basis where the zero modes with respect to ρ become the diagonal elements,

$$\begin{aligned} P^\dagger X(\theta)|_{k\text{-th}} P &= S^\dagger T^\dagger X(\theta)|_{k\text{-th}}\, T S \\ &= S^\dagger \left(\sum_{p=-\infty}^{\infty} \sum_{q=-M}^{M} X_{(pN+q,k)}\, \tau^{p+\frac{q}{N}}\, \lambda^{-k(q+1)} U^{-k} V^q \right) S \\ &= \sum_{p=-\infty}^{\infty} \sum_{q=-M}^{M} X_{(pN+q,k)}\, \tau^{p+\frac{q}{N}}\, \lambda^{-k(q+1)} V^k U^q \end{aligned}$$

$$
\equiv \begin{pmatrix}
\overbrace{0 \quad \cdots \quad 0}^{k} & X_{1\,k+1}(\theta) & & & & \\
 & 0 & X_{2\,k+2}(\theta) & & & \\
 & & \ddots & \ddots & & \\
 & & & 0 & X_{N-k\,N}(\theta) & \\
X_{N-k+1\,1}(\theta) & & & & 0 & \\
 \ddots & & & & \vdots & \\
\underbrace{0 \quad \cdots \quad 0}_{k-1} \; X_{N\,k}(\theta) & 0 & & & 0 &
\end{pmatrix} . \tag{4.24}
$$

The non-zero matrix elements,

$$
\left.\begin{array}{l} X_{a\,k+a}(\theta) \quad (a = 1, \ldots, N-k) \\ X_{a\,k+a-N}(\theta)\;(a = N-k+1, \ldots, N) \end{array}\right\} = \sum_{p=-\infty}^{\infty} \sum_{q=-M}^{M} X_{(pN+q,k)} e^{i2(p+\frac{q}{N})\theta} e^{i\frac{2k(q-1)\pi}{N}} \; e^{i\frac{4(a-1)\pi}{N}q}
$$
$$
= \sum_{n=-\infty}^{\infty} X_{(n,k)}\, e^{i2\frac{n}{N}(\theta+2(a-1)\pi)}\, e^{i2\pi\frac{n-1}{N}k} \tag{4.25}
$$

satisfy the following boundary conditions,

$$
\begin{aligned}
X_{a\,k+a}(\theta+2\pi) &= X_{a+1\,k+a+1}(\theta)\,, \qquad (a = 1, \ldots, N-k-1) && (4.26)\\
X_{N-k\,N}(\theta+2\pi) &= X_{N-k+1\,1}(\theta)\,, && (4.27)\\
X_{a\,k+a-N}(\theta+2\pi) &= X_{a+1\,k+a-N+1}(\theta)\,, \qquad (a = N-k+1, \ldots, N-1) && (4.28)\\
X_{N\,k}(\theta+2\pi) &= X_{1\,k+1}(\theta)\,. && (4.29)
\end{aligned}
$$

In the case of $-k$-th Fourier modes ($k > 0$), the matrix is given by the hermitean conjugation of eq. (4.24).

Furthermore, by using the unitary matrix P (4.14), the matrix representation of the linear function ρ (3.5) is transformed as follows,

$$
\begin{aligned}
P^\dagger \rho P &= S^\dagger T^\dagger \rho T S \\
&= S^\dagger(-2\pi i)\partial_\theta \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} S \\
&= -2\pi i \partial_\theta \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} .
\end{aligned} \tag{4.30}
$$

Thus in the basis where the zero modes w.r.t. ρ are diagonalized, *the matrix representation of the linear function ρ is proportional to the derivative $-i\partial_\theta$ times the unit matrix.*[7] On the other hand, in the original basis, the matrix representation (3.5) is not proportional to the unit matrix. Henceforth, all matrices are represented in such basis as the zero modes w.r.t. ρ are diagonalized and we rewrite $P^\dagger X(\theta)P$ $(X = Y, X^k, A)$ and $P^\dagger \rho P$ to $X(\theta)$ and ρ, respectively. Then, the matrix representations of X^9, X^k and A are given by

$$X^9 \to -2\pi i w L\partial_\theta \begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} + Y(\theta)\,, \tag{4.31}$$

$$X^k \to X^k(\theta)\,, \tag{4.32}$$

$$A \to A(\theta)\,. \tag{4.33}$$

From eqs. (4.21)–(4.22) and (4.26)–(4.29), we find that $Y(\theta), X^k(\theta)$ and $A(\theta)$ satisfy the boundary conditions,

$$Y(\theta+2\pi) = VY(\theta)V^\dagger\,, \tag{4.34}$$

$$X^k(\theta+2\pi) = VX^k(\theta)V^\dagger\,, \tag{4.35}$$

$$A(\theta+2\pi) = VA(\theta)V^\dagger\,. \tag{4.36}$$

In ref. [12], the boundary conditions were assumed, while they are derivable in our case.

Finally, we show that after the consistent truncation, the action of the light-cone wrapped supermembrane on $R^{10}\times S^1$ agrees with matrix string theory [12, 14]. In such a truncation, the functions X^9, X^k, A of σ and ρ are represented by the matrices (4.31)–(4.33) and the Poisson bracket and the double integral are represented as follows,

$$\{\,\cdot\,,\,\cdot\,\} \to -i\frac{N}{4\pi}[\,\cdot\,,\,\cdot\,]\,, \tag{4.37}$$

$$\int_0^{2\pi} d\sigma d\rho \to \frac{2\pi}{N}\int_0^{2\pi} d\theta\,\mathrm{Tr}. \tag{4.38}$$

From these results, the action (4.1) in the case of the single wrapping is mapped to

$$S_{\mathrm{MS}} = \frac{LT}{2}\int d\tau \frac{\pi N}{2}\int_0^{2\pi} d\theta\,\mathrm{Tr}\left[(F_{\tau\theta})^2 + (D_\tau X^k)^2 - (D_\theta X^k)^2 + \frac{1}{2(2\pi L)^2}\left[X^k, X^l\right]^2\right], \tag{4.39}$$

$$F_{\tau\theta} = \frac{2}{N}\partial_\tau Y - \partial_\theta A + i\frac{1}{2\pi L}[A, Y]\,, \tag{4.40}$$

[7] Precisely speaking, this statement is not always correct because the transformation matrix P (4.14) has an ambiguity of the overall phase $e^{i\alpha\theta}$. However, even if we have included such a phase factor in eq. (4.14), the matrix representation of ρ in the transformed basis is proportional to the unit matrix since the additional term is proportional to α times the unit matrix. And such an extra term does not affect matrix string theory (4.43).

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$$D_\tau X^k = \frac{2}{N}\partial_\tau X^k + i\frac{1}{2\pi L}[A, X^k]\,, \tag{4.41}$$
$$D_\theta X^k = \partial_\theta X^k + i\frac{1}{2\pi L}[Y, X^k]\,. \tag{4.42}$$

By rescaling $\tau \to (2/N)\,\tau$, we obtain

$$S_{\rm MS} = \frac{\pi LT}{2}\int d\tau \int_0^{2\pi} d\theta\, {\rm Tr}\left[(F_{\tau\theta})^2 + (D_\tau X^k)^2 - (D_\theta X^k)^2 + \frac{g^2}{2}\left[X^k, X^l\right]^2\right], \tag{4.43}$$
$$F_{\tau\theta} = \partial_\tau Y - \partial_\theta A + ig[A, Y]\,, \tag{4.44}$$
$$D_\tau X^k = \partial_\tau X^k + ig[A, X^k]\,, \tag{4.45}$$
$$D_\theta X^k = \partial_\theta X^k + ig[Y, X^k]\,, \tag{4.46}$$

where $g = 1/(2\pi L)$. The fields $Y(\theta), X^k(\theta)$ and $A(\theta)$ satisfy the boundary conditions (4.34)–(4.36). This action is just a bosonic part of matrix string theory, i.e., 1+1-dimensional U(N) super Yang-Mills theory. Thus we have obtained the matrix string theory having the boundary conditions for the matrix variables corresponding to the wrapped supermembrane.

5. Multi-wrapped membranes in matrix string theory

In this section, we consider the matrix regularization of the multi-wrapped supermembrane on $R^{10} \times S^1$ in the light-cone gauge. Similarly to eqs. (4.43)–(4.46), the multi-wrapped supermembrane action is matrix-regularized as

$$S^{(w)}_{\rm MS} = \frac{\pi LT}{2}\int d\tau \int_0^{2\pi} d\theta\, {\rm Tr}\left[(F_{\tau\theta})^2 + (D_\tau X^k)^2 - (D_\theta X^k)^2 + \frac{g^2}{2}\left[X^k, X^l\right]^2\right], \tag{5.1}$$
$$F_{\tau\theta} = \partial_\tau Y - w\partial_\theta A + ig\,[A, Y]\,, \tag{5.2}$$
$$D_\tau X^k = \partial_\tau X^k + ig\,[A, X^k]\,, \tag{5.3}$$
$$D_\theta X^k = w\partial_\theta X^k + ig\,[Y, X^k]\,, \tag{5.4}$$

where $g = 1/(2\pi L)$ and w is the wrapping number.[8] By rescaling $\tau \to \tau/w$, we obtain

$$S^{(w)}_{\rm MS} = \frac{w\pi LT}{2}\int d\tau \int_0^{2\pi} d\theta {\rm Tr}\left[\left(F^{(w)}_{\tau\theta}\right)^2 + \left(D^{(w)}_\tau X^k\right)^2 - \left(D^{(w)}_\theta X^k\right)^2 + \frac{g^{(w)2}}{2}\left[X^k, X^l\right]^2\right], \tag{5.5}$$
$$F^{(w)}_{\tau\theta} = \partial_\tau Y - \partial_\theta A + ig^{(w)}[A, Y]\,, \tag{5.6}$$
$$D^{(w)}_\tau X^k = \partial_\tau X^k + ig^{(w)}[A, X^k]\,, \tag{5.7}$$
$$D^{(w)}_\theta X^k = \partial_\theta X^k + ig^{(w)}[Y, X^k]\,, \tag{5.8}$$

where $g^{(w)} = 1/(2\pi wL)$. In order to see the physical meaning, we consider the double-dimensional reduction of this action. Classically, this is to remove the off-diagonal matrix

[8] In this section, for simplicity, we consider $w > 0$ case only.

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elements by hand. Then, we obtain the discretized action of a ten-dimensional superstring with w times the minimal string tension. However, such objects cannot be incorporated in type-IIA superstring theory. One possible interpretation is to regard the w-wrapped supermembrane as w fundamental type-IIA superstrings rather than as a ten-dimensional superstring with w times the minimal string tension [19]. Note that after the double-dimensional reduction, the w-dependence in action (5.5) through $g^{(w)}$ is disappeared.

We also can embed the multi-wrapped supermembrane into matrix string theory, i.e., the single-wrapped supermembrane in the matrix-regularized form. By rescaling $\theta \to w\theta$ in eq. (5.1), we obtain

$$\begin{aligned}
S_{MS}^{(w)} &= w \int d\tau \int_0^{\frac{2\pi}{w}} d\theta \, \mathcal{L}(Y(w\theta), X^k(w\theta), A(w\theta)) \,, && (5.9)\\
\mathcal{L}(Y(\theta), X^k(\theta), A(\theta)) &= \frac{\pi LT}{2} \mathrm{Tr} \left[(F_{\tau\theta})^2 + \left(D_\tau X^k\right)^2 - \left(D_\theta X^k\right)^2 + \frac{g^2}{2}\left[X^k, X^l\right]^2 \right], && (5.10)\\
F_{\tau\theta} &= \partial_\tau Y - \partial_\theta A + ig[A, Y] \,, && (5.11)\\
D_\tau X^k &= \partial_\tau X^k + ig[A, X^k] \,, && (5.12)\\
D_\theta X^k &= \partial_\theta X^k + ig[Y, X^k] \,. && (5.13)
\end{aligned}$$

Note that the lagrangian $\mathcal{L}(Y(\theta), X^k(\theta), A(\theta))$ is that of matrix string theory, i.e., matrix-regularized lagrangian of the single-wrapped supermembrane. This action is rewritten as follows,

$$\begin{aligned}
S_{\mathrm{MS}}^{(w)} &= \int d\tau \int_0^{\frac{2\pi}{w}} d\theta \, \mathcal{L}(Y(w\theta), X^k(w\theta), A(w\theta)) + \\
&\quad + \int d\tau \int_{\frac{2\pi}{w}}^{\frac{4\pi}{w}} d\theta \, \mathcal{L}(Y(w\theta - 2\pi), X^k(w\theta - 2\pi), A(w\theta - 2\pi)) + \\
&\quad \vdots \\
&\quad + \int d\tau \int_{\frac{w-1}{w}2\pi}^{2\pi} d\theta \mathcal{L}(Y(w\theta - (w-1)2\pi), X^k(w\theta - (w-1)2\pi), A(w\theta - (w-1)2\pi)) \\
&= \int d\tau \int_0^{\frac{2\pi}{w}} d\theta \, \mathcal{L}(Y(w\theta), X^k(w\theta), A(w\theta)) + \\
&\quad + \int d\tau \int_{\frac{2\pi}{w}}^{\frac{4\pi}{w}} d\theta \, \mathcal{L}(V^\dagger Y(w\theta) V, V^\dagger X^k(w\theta) A, V^\dagger A(w\theta) V) + \\
&\quad \vdots \\
&\quad + \int d\tau \int_{\frac{w-1}{w}2\pi}^{2\pi} d\theta \mathcal{L}((V^{w-1})^\dagger Y(w\theta) V^{w-1}, (V^{w-1})^\dagger X^k(w\theta) V^{w-1}, (V^{w-1})^\dagger A(w\theta) V^{w-1}),
\end{aligned}$$

where we have used the boundary conditions (4.34)–(4.36). Due to gauge invariance, we have

$$\mathcal{L}(Y(\theta), X^k(\theta), A(\theta)) = \mathcal{L}(V^\dagger Y(\theta) V, V^\dagger X^k(\theta) V, V^\dagger A(\theta) V) \,. \tag{5.14}$$

Then we obtain

$$
\begin{aligned}
S_{MS}^{(w)} &= \int d\tau \int_0^{2\pi} d\theta \, \mathcal{L}(Y(w\theta), X^k(w\theta), A(w\theta)) \\
&= \int d\tau \int_0^{2\pi} d\theta \, \mathcal{L}(Y^{(w)}(\theta), X^{k(w)}(\theta), A^{(w)}(\theta)) \,, \qquad (5.15)
\end{aligned}
$$

where $X^{(w)}(\theta) \equiv X(w\theta)$ $(X = Y, X^k, A)$. Thus we have succeeded in naturally embedding the multi-wrapped supermembrane into matrix string theory.

6. Conclusion

In this paper, we have given a concrete matrix representation of the infinite dimensional Lie algebra [14] to obtain matrix string theory via matrix regularization for the wrapped supermembrane on $R^{10} \times S^1$ in the light-cone gauge. We have explicitly given the correspondence of matrix string with the wrapped supermembrane. That is, in eqs. (4.20) and (4.25), the matrix elements in matrix string theory are determined completely by the Fourier coefficients in the wrapped supermembrane theory. Furthermore, eqs. (4.20) and (4.25) determine the boundary conditions for the matrix variables in matrix string theory. We should notice that we have never used the standard Seiberg and Sen's arguments and string dualities in obtaining the matrix string theory in this paper. Thus, this method gives support to the string dualities and the recovery of eleven dimensional Lorentz invariance in the large-N limit.

Note added: while finishing the manuscript, a complementary paper [20] appeared in the e-print archive, where matrix string theory is derived with the string dualities and the 9-11 flip. Furthermore, see ref. [21] for a different approach to the wrapped supermembrane.

Acknowledgments

We would like to thank N. Kitsunezaki for useful discussion. This work is supported in part by MEXT Grant-in-Aid for the Scientific Research #13135212 (S.U.) and JSPS Grant-in-Aid for the Scientific Research (B)(2) #14340072 (S.Y.).

References

[1] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Supermembranes and eleven-dimensional supergravity*, *Phys. Lett.* **B 189** (1987) 75.

[2] J. Hoppe, *Quantum theory of a relativistic membrane*, M.I.T. Ph.D. thesis, 1982

[3] B. de Wit, J. Hoppe and H. Nicolai, *On the quantum mechanics of supermembranes*, *Nucl. Phys.* **B 305** (1988) 545.

[4] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[5] L. Susskind, *Another conjecture about M(atrix) theory*, hep-th/9704080.

JHEP07(2004)043

[6] L. Motl, *Proposals on nonperturbative superstring interactions*, hep-th/9701025.

[7] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Matrix string theory*, *Nucl. Phys.* **B 500** (1997) 43 [hep-th/9703030].

[8] N. Seiberg, *Why is the matrix model correct?*, *Phys. Rev. Lett.* **79** (1997) 3577 [hep-th/9710009].

[9] A. Sen, *D0-branes on T^n and matrix theory*, *Adv. Theor. Math. Phys.* **2** (1998) 51 [hep-th/9709220].

[10] M.J. Duff, P.S. Howe, T. Inami and K.S. Stelle, *Superstrings in $D = 10$ from supermembranes in $D = 11$*, *Phys. Lett.* **B 191** (1987) 70.

[11] J.G. Russo, *Supermembrane dynamics from multiple interacting strings*, *Nucl. Phys.* **B 492** (1997) 205 [hep-th/9610018].

[12] Y. Sekino and T. Yoneya, *From supermembrane to matrix string*, *Nucl. Phys.* **B 619** (2001) 22 [hep-th/0108176].

[13] S. Uehara and S. Yamada, *On the strong coupling region in quantum matrix string theory*, *J. High Energy Phys.* **09** (2002) 019 [hep-th/0207209]; *On the quantum matrix string*, hep-th/0210261.

[14] M. Cederwall, *Open and winding membranes, affine matrix theory and matrix string theory*, *J. High Energy Phys.* **12** (2002) 005 [hep-th/0210152].

[15] H. Shimada, *Membrane topology and matrix regularization*, *Nucl. Phys.* **B 685** (2004) 297 [hep-th/0307058].

[16] D.B. Fairlie, P. Fletcher and C.K. Zachos, *Trigonometric structure constants for new infinite algebras*, *Phys. Lett.* **B 218** (1989) 203;
D.B. Fairlie and C.K. Zachos, *Infinite dimensional algebras, sine brackets and* SU(∞), *Phys. Lett.* **B 224** (1989) 101.

[17] J. Fuchs, *Affine Lie algebras and quantum groups: an introduction, with applications in conformal field theory*, SPIRES entry.

[18] S. Uehara and S. Yamada, *Comments on the global constraints in light-cone string and membrane theories*, *J. High Energy Phys.* **12** (2002) 041 [hep-th/0212048].

[19] K. Becker, M. Becker and A. Strominger, *Five-branes, membranes and nonperturbative string theory*, *Nucl. Phys.* **B 456** (1995) 130 [hep-th/9507158].

[20] M. Hayakawa and N. Ishibashi, *Perturbative dynamics of matrix string for the membrane*, *J. High Energy Phys.* **04** (2004) 047 [hep-th/0401227].

[21] J. Dai and Y.S. Wu, *Quiver matrix mechanics for IIB string theory, I. Wrapping membranes and emergent dimension*, *Nucl. Phys.* **B 684** (2004) 75 [hep-th/0312028].

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